# The Unique, Well Posed Reduced System for Atmospheric Flows: Robustness In The Presence Of Small Scale Surface Irregularities

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Dedicated to my mentor and colleague, Heinz-Otto Kreiss

**Abstract**

Numerical analysis requires that a number of derivatives of the continuum solution of any differential system of equations exist in order that the numerical approximations of the derivatives of that system are ensured to have sufficiently small truncation errors. If a hyperbolic partial differential system of equations also contains multiple time scales (as is the case for the atmospheric equations of motion) and it is the goal to accurately compute the component of the solution that evolves on the time scale of the advective terms (the component with the majority of the energy), then additional restrictions are required of the derivatives. For the initial value problem, deriving higher order time derivatives of the continuum solution using the Bounded Derivative Theory (Kreiss 1979 , Kreiss 1980) will lead to spatial elliptic constraints that must be satisfied to ensure the ensuing solution will evolve on a space and time scale of order unity in the scaled system. For the initial/boundary problem, the boundary conditions for the elliptic constraints must be derived from the well posed boundary conditions of the original hyperbolic system that ensure that

the ensuing solution in the limted area evolves on space and time scales of order unity. If these requirements are met, then the $L_2$ energy estimates of the solution and a number of its spatial and temporal derivatives are independent of the fast time scales and ensure that the resulting limit system (the reduced system) as the fast scales approach infinity will automatically be well posed. In this manuscript the reduced system for large scale atmospheric flows will be introduced and a special property of the corresponding elliptic equation for the vertical component of the velocity will be discussed. In particular, the solution of that elliptic equation is not sensitive to small scale perturbations at the lower boundary so it can be used all of the way to the surface.

# 1 Introduction

The five time dependent partial differential equations for entropy(1), mass(1) and momentum(3) describe the evolution of many different kinds of fluids. The natural mathematical question is how does one understand the behavior of a particular kind of fluid exhibiting a particular kind of behavior. This question has been answered by introducing a scale analysis of the particular kind of flow of the fluid, i.e., by introducing characteristic variables that describe the typical values of the independent and dependent variables of that flow and then making a simple change of variables using those values. This technique is helpful in identifying the relative sizes of the individual terms in any given equation and has been used in many scientific areas including meteorology (Charney 1948), oceanography (Browning, Holland, and Worley 1989) and plasma physics (Browning and Holzer 1992). However, that scale analysis does not ensure that the given flow will continue to evolve in time with the same chosen scales of motion.

Kreiss (Kreiss 1979 , Kreiss 1980) introduced the Bounded Derivative Theory

(BDT) for scaled (nondimensional) hyperbolic systems of equations with the advective terms of order unity (space and time scales of order of the advective terms) and off diagonal terms much greater than unity contributing to high frequency (fast time scale) components of the solution. To ensure that the ensuing solution would evolve on the order of the advective terms, i.e., of order unity, the mathematical $L_2$ energy method applied to the solution and its space and time derivatives must yield norms of those functions on the order of unity for a time period on the order of the time scale of the advective terms. As these estimates of the solution and its space and time derivatives are independent of the fast (high frequency) time scales, the estimates hold as the size of the large terms increase to infinity. Thus the system that represents this limit, the reduced system, also satisfies these so estimates is automatically well posed and accurately describes the motion of interest.

A review of the BDT for the atmospheric equations for large scale flows in the midlatitudes is presented in Section 2. The scaled system from Browning and Kreiss (Browning and Kreiss 1986) is reproduced to reveal how the scaling produces a nondimensional system with large off diagonal terms. A simple example is used to show how the space and time derivatives become coupled so that both must be estimated to ensure a slowly evolving solution. Section 3 introduces the reduced system for large scale midlatitude flows, although it has been shown that the reduced system also accurately describes mesoscale flows (Browning and Kreiss 2002). Numerical examples are presented in Section 4. The examples use a heating function that is large scale in space and time to generate an evolving large scale solution. The solutions from the model based on the multiscale system and the model based on the reduced system are compared. In contrast to Richardson's equation for the vertical velocity in the primitive (hydrostatic) equations, it is demonstrated that the solution of the elliptic equation for the vertical velocity in the reduced system is not sensitive to small scale noise at the lower boundary.

## 2 Bounded Derivative Theory Review

To determine the relative size of individual terms in a given equation of the partial differential system that describes large scale atmospheric motions, a simple change of variables is used. The characteristic scales of the independent and dependent variables describing the motion are used for this purpose, e.g., a horizontal length scale L = 1000 km, a depth scale of D = 10 km, a time scale T = 86400 sec (1 day), a horizontal velocity scale U = 10 m/s and a vertical velocity scale W = .01 m/s. The pressure and density are scaled as perturbations of a mean state in hydrostatic equilibrium. For large scale motion in the atmosphere this leads to the following scaled (nondimensional) system of equations (Browning and Kreiss 1986):

$$\frac{ds}{dt} - \tilde{s}(w - H) = 0, \tag{2.1a}$$

$$\frac{du}{dt} + \epsilon^{-1}(\rho_0^{-1}p_x - fv) = 0, \tag{2.1b}$$

$$\frac{dv}{dt} + \epsilon^{-1}(\rho_0^{-1}p_y + fu) = 0, \tag{2.1c}$$

$$\frac{dw}{dt} + \alpha\epsilon^{-6}(\rho_0^{-1}p_z + \tilde{p}p + gs) = 0, \tag{2.1d}$$

$$\frac{dp}{dt} + \epsilon^{-1}wp_{0z} + \epsilon^{-2}\gamma p_0(u_x + v_y + \epsilon w_z) = 0, \tag{2.1e}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + \epsilon w\partial/\partial z$. The nondimensional dependent variables $s, u, v, w, p$ are the reciprocal of entropy (hereafter referred to as entropy for brevity), velocity components, and pressure perturbation from the mean, respectively. The dimensionless functions $\rho_0(z)$ and $p_0(z)$ are the mean hydrostatic state values of the density and pressure, $s_0 = \rho_0 p_0^{-1/\gamma}$, $\tilde{s} = 10s_{0z}/s_0$,

$\tilde{p} = -p_{0z}/(\gamma\rho_0 p_0)$ and $f(y)$ is the Coriolis term. The nondimensional constant $g$ is the gravitational constant and $\gamma = 1.4$. The dimensionless function $H(x, y, z, t)$ is essentially the sum of all heating and cooling sources. Inverse powers of $\epsilon = 1/10$ represent large terms. (Note that the original dimensional equations can be obtained by setting $\epsilon = 1$.) For the original scaling $\alpha = 1$, but for the multiscale system it is $\alpha = (D/L)^2$ which is $\epsilon^4 = 10^{-4}$. The latter value of $\alpha$ has been proved mathematically to reproduce the slowly evolving in time solution of (2.1) with $\alpha = 1$ to at least two digits of accuracy. Because there are five time dependent equations there are five different frequencies: one associated with advective motions, two with inertial/gravity waves and two with sound waves (Browning and Kreiss 1985). The BDT theory was developed to be able to select the five initial conditions so that the components of the solution that are associated with the latter four frequencies would remain small for a given period of time.

The first such scaling was performed by Charney without the mean hydrostatic state removed (Charney 1947, Charney 1948) and subsequently by Browning and Kreiss (Browning and Kreiss 1986) with the mean hydrostatic state removed. Charney discovered that there were two large terms that were 8 orders of magnitude larger (6 orders with the mean state removed) than the remaining terms in the time dependent equation for the vertical component of velocity, dw/dt, that were impossible to compute accurately using numerical methods of the time. Instead of doing so, the two terms, the vertical pressure gradient and the gravitational term, were set to be equal. This equality between these two terms is called hydrostatic balance and leads to a modification of the system called the primitive equations. The resulting columnar integral equation at each horizontal point for the vertical velocity is called Richardson's equation.

A scale analysis does not by itself prove that the motion will evolve as described by the characteristic scales of motion. This must be done by mathematics using the theory of hyperbolic systems of equations in conjunction with the Bounded Derivative Theory (Kreiss 1979, Kreiss 1980). To determine the subsequent motion of such a system requires estimates of the ensuing in time spatial and temporal derivatives. The estimates for these derivatives are determined by differentiating the equations with respect to space or time as appropriate and then using the mathematical $L_2$ energy method to estimate the norms of the derivatives at a later time to ensure that the solution will continue to evolve on the slow time scale. An important detail in such arguments can be considered using the so called Kreiss equation

$$u_t = a(x,t)u_x. \tag{2.2}$$

To estimate the ensuing time derivative of a solution of this equation differentiate the equation with respect to time

$$(u_t)_t = a(x,t)(u_t)_x + a_t(x,t)u_x. \tag{2.3}$$

Note that $u_t$ satisfies the same equation as $u$ with the exception of one term. That term couples the space and time derivative terms. Thus in the BDT the energy estimates must show that both the higher order spatial and temporal derivatives evolve as specified in the scaling, namely, with the space scale L = 1000 km and the time scale on the order of T = 1 day, or the solution will not continue to evolve in the chosen manner. This requirement precludes the primitive equation solution from evolving correctly because the columnar equation for the vertical component of the velocity can change discontinuously from horizontal point to point because of switches in the heating parameterizations. Those discontinuities violate the spatial derivative estimates required by the BDT. They also require unrealistically large dissipation because they inject energy into the smallest scales in a numerical model. That dissipation reduces the numerical accuracy of a numerical method by orders of magnitude (Browning, Hack, and Swarztrauber 1989, Browning and Kreiss 1994 ). Note that the initial-boundary value problem for the primitive equations is not well posed (Oliger and Sundström 1978) and that also indicates that it is not the correct reduced system.

Using the BDT to initialize the unmodified hyperbolic Euler equations with the appropriate space and time derivatives ensures that the evolution of the solution on the chosen scales will require no dissipation and minimal numerical accuracy as will be shown in the numerical examples to follow. As has been discussed before (Browning and Kreiss 2002), the elliptic initialization constraints can be used in conjunction with a time dependent equation for the vertical component of vorticity to form an automatically well posed system that accurately describes the evolution of the large scale motion.

# 3 Reduced System

In this section the reduced system for large scale atmospheric motions will be described. As mentioned previously, a time dependent equation for the vertical component of vorticity (the only variable that can be used globally for the time dependent slowly evolving variable in time and space for all scales of motion) is added to the initialization constraints for the multiscale system. The time dependent equation for the vertical component of vorticity $\zeta = -u_y + v_x$ can be derived by appropriately cross differentiating equations (2.1a) and (2.1b):

$$\frac{d\zeta}{dt} + v_z w_x - u_z w_y + (f + \delta)\zeta + f_y v = 0, \tag{3.1}$$

where $\delta = u_x + v_y$ is the horizontal component of divergence. The elliptic initialization constraints for $s$ and the vertical velocity $w$ are (Browning and Kreiss 2002)

$$\bigtriangledown^2 s = -\{\rho_0[f\zeta - f_y u + 2(u_x v_y + u_y v_x)]\}_z/(\rho_0 g), \tag{3.2}$$

$$\bigtriangledown^2 w + f^2(g\tilde{s})^{-1}[w_{zz} + \rho_0 z(\rho_0)^{-1} w_z] = \bigtriangledown^2 H - (g\tilde{s})^{-1} R_1, \tag{3.3}$$

$$R_1 = -gC_2 - fC_1\ \rho_0, \tag{3.4}$$

$$C_1 = u_z(\rho_0\zeta)_x + v_z(\rho_0\zeta)_y, \tag{3.5}$$

$$C_2 = u_{xx}s_x + 2u_x s_{xx} + u_{yy}s_x + 2u_y s_{xy} \tag{3.6}$$

$$+ v_{xx}sy + 2v_x s_{xy} + v_{yy}s_y + 2v_y syy, \tag{3.7}$$

where $C_1$ and $C_2$ are the commutators derived previously (Browning and Kreiss 2002). The quantity between French braces in the equation for $s$ is essentially just the vertical derivative of the right hand side of the nonlinear balance equation, i.e., the equation is derived by using the two dominant terms of hydrostatic balance from equation (2.1d). (The additional $\tilde{p}$ term can be added in a similar manner.) The horizontal smoothing of the right-hand-side of the balance equation is retained in the elliptic equation for $s$, but there is no smoothing of the vertical derivative. However, in the equation for $w$, there is vertical smoothing and that is what results in the well posedness of the reduced system. By using the equation for s instead of the one for p, the mean of s is not required as only derivatives of s appear in the right-hand-side for w. Note that we have neglected a number of terms of order $\epsilon$ in the derivation of the equation for w to simplify the presentation. If required, they can be added by a simple iterative method.

The equation for $w$ has several very special properties, namely, that small scale perturbations of the lower boundary condition have only a minor impact on the solution while larger scale perturbations do. This is physically important in both cases, e.g., the so called lake effect on large scale storms. Thus the equation for $w$ can be used all of the way to the surface without the need for the ad hoc discontinuous boundary layer parameterization to artificially slow down the unrealistic growth of the velocity at the surface when using Richardson's equation (Sylvie Gravel, personal communication). Note that the equation for $w$ is similar to the quasi-geostrophic $\omega$ equation (Charney 1947,Charney 1948).

The horizontal divergence in the reduced system is given by the balance between the large terms in equation (2.1e)

$$\delta = -[w_z + wp_{0z}(\gamma p_0)^{-1}]. \tag{3.8}$$

Given the vorticity $\zeta$ and the divergence $\delta$, the horizontal components of velocity must be computed from the Helmholtz equations

$$\bigtriangledown^2 u = -\zeta_y + \delta_x, \tag{3.9}$$

$$\bigtriangledown^2 v = \zeta_x + \delta_y, \tag{3.10}$$

in order to connect these constraints to the well posed boundary conditions for the hyperbolic system (2.1) that ensure a slowly evolving solution in a limited area.

# 4 Numerical Examples

The details of the numerical approximation of the multiscale system in a channel 2000 km square and 12 km high have been presented earlier (Browning and

Kreiss 2002) so here we just summarize that method. The multiscale equations were approximated by the leapfrog method in space and time. The spatial derivatives needed no special treatment in $x$ because the solution was periodic in that direction. At the north and south wall boundaries the $y$ component of velocity $v = 0$ so the boundaries were treated with inflow/outflow conditions with the $y$ component of the velocity in that treatment set to 0. Similarly for $w$ at the bottom and top boundaries.

For the reduced system, the fourth order Runge-Kutta method in time and second order centered differences in space are used for the vorticity equation with the diagnostic quantities determined at each stage using the elliptic equations for $(u, v)$, $s$, $w$ then $(u, v)$ again in that order. No special treatment is needed for the vorticity equation at either the north or south boundaries. Note that in equation (2.1a) for $s$, at the boundaries at the bottom and top of the channel $w = 0$. Thus, if the initial value of s is zero and there is no heating on those boundaries, $s$ will remain identically zero there even as the horizontal velocities become nonzero. If there is initial horizontal velocity on those boundaries, $s$ will be horizontally advected with the heating (if any) acting as a forcing term. In either case this provides the variable $s$ at the bottom and top of the channel. In the equation for $u$ the boundary condition $-u_y = \zeta$ is used at the north and south boundaries, while $v$ is identically zero there.

To simplify the presentation in the numerical results to follow, $\tilde{p}$ and $p_{0z}$ are neglected (this has an impact on the physical solution, but not the mathematics as these terms are anti-symmetric). The initial condition is $\zeta(x, y, z, 0) = 0$ and the heating function is

$$H = H_1 H_2, \tag{4.1}$$

$$H_1 = .01 \sin^4(\pi y/L_1) \sin^2(\pi z/z_T), \tag{4.2}$$

$$H_2 = t_1 \sin(2\pi s_1/L_1) H_1, \tag{4.3}$$

where $L_1 = 6000$ km is the size of the square horizontal domain, $z_T = 12$ km is the height at the top of the channel, the time factor $t_1 = 1 - \exp(-t/86400)$, the shift factor $s_1 = x - u_0 t$ and $u_0 = 10$ m/s. Note that the heating is $O(1)$ in scaled terms, i.e., the magnitude is equal to the scaling value of W, the x derivative is of size $W 2\pi/L_1 \approx L^{-1} W$ and the storm is essentially the height of the entire atmosphere. This heating consists of large scale warm and cold air masses moving eastward at a velocity of 10 m/s. Note that the heating is 0 at the bottom and top of the channel, starts out slowly until essentially a maximum is reached at 2 days. The results that follow will be shown at 4 days (two days after maximum heating has been achieved). The grid sizes for models are $\Delta x = \Delta y = 100$ km, $\Delta z = 1$ km, $\Delta t = 40$ sec for the multiscale model and $\Delta t = 1800$ sec for the reduced model.

Figure 1 shows the pressure perturbation from the multiscale modelas a function of time at the three horizontal grid points shown at the top of the plot at a height of 3 km. Although the multiscale system has both low and high frequencies present, they are clearly not activated as expected if the $L_2$ norms of the space and time derivatives of the nondimensional heating term are on the order of the advective terms.

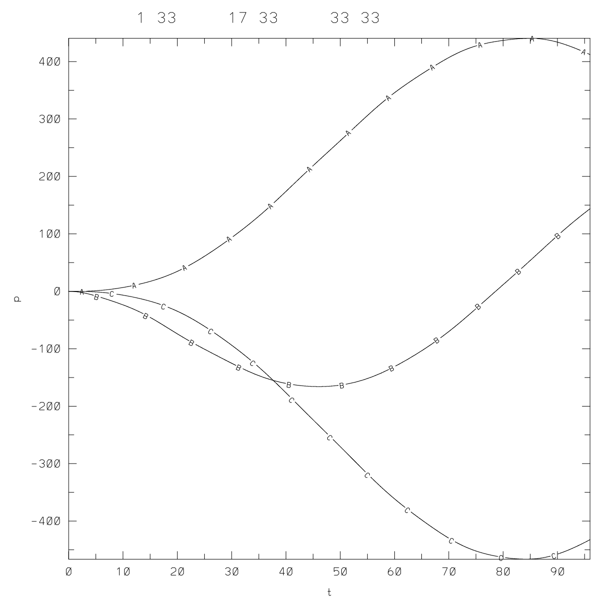
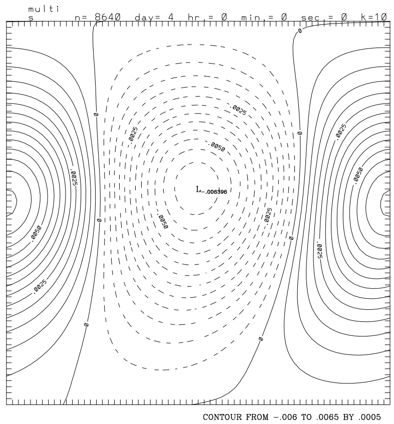
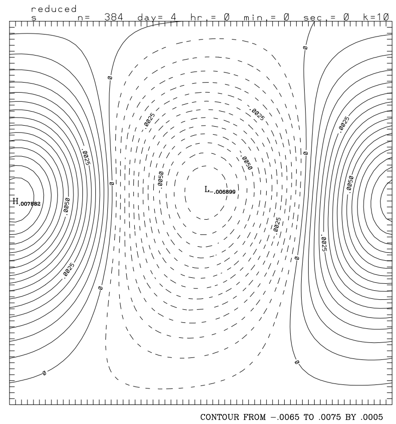



Figure 1: Multiscale pressure perturbation as a function of time at the (x,y) points at the top of the plot at z = 3 km

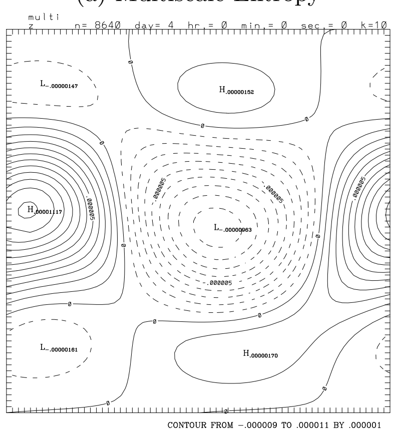
Note that the multiscale and reduced systems are derived in completely different manners. The multiscale hyperbolic system was mathematically proved to describe the slowly evolving in time large scale atmospheric motions by a continuum modification of system (2.1). The reduced system is derived from the initialization constraints for (2.1) with the addition of a time dependent equation for the vorticity. Because both independently are expected to describe the same large scale slowly evolving solution a comparison of the solutions from models based on the two different systems is of interest. Figure 2 on the following page compares the results from the numerical model based on the multiscale system (left hand side) and the numerical model based on the reduced system (right hand side) for the variables shown at $z = 9$ km and $t = 4$ days. As expected from the BDT, the solutions from the two models are quite similar. Although only one level is shown, the relative $l_2$ errors are 9.1%, 8.9% and 8.2% for the horizontal divergence $\delta$, the vertical component of velocity w and the vertical component of vorticity $\zeta$, respectively. As a number of terms of order $\epsilon$ have been neglected, e.g., the term $\delta\zeta$ in the vorticity equation in the derivation of the equation for w, these errors are completely reasonable. The lower level horizontal velocities in a primitive equation model grow unrealistically large in a few days and require an ad hoc boundary layer drag/dissipation to artificially slow down that growth (Sylvie Gravel, personal communication). Note that nether the multiscale or reduced model include any dissipation.



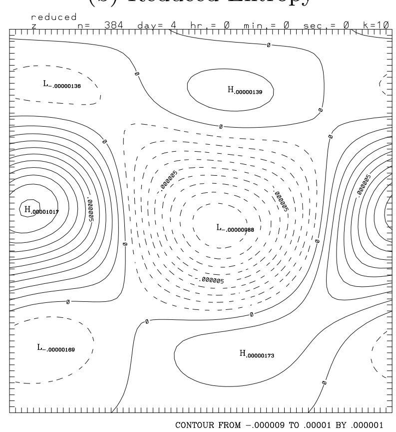

(a) Multiscale Entropy



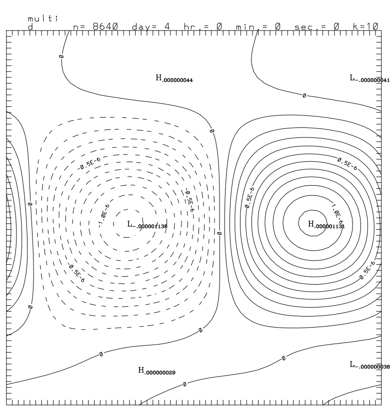

(b) Reduced Entropy



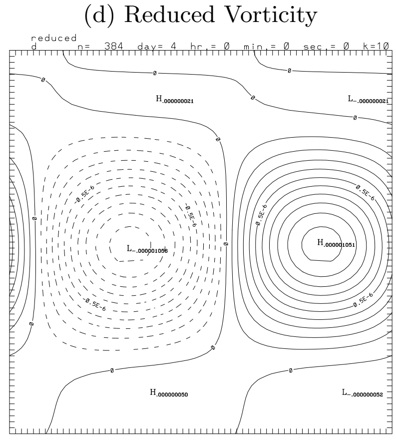

(c) Multiscale Vorticity




(d) Reduced Vorticity




(e) Multiscale Divergence




(f) Reduced Divergence

Figure 2: Comparison of multiscale model variables (left column) and reduced system model variables (right hand column) at z = 9 km and t = 4 days

There is a very important property of the elliptic equation for $w$ that will be demonstrated next. Small scale noise at the lower boundary of that equation, e.g., noise caused by individual trees or rocks or small scale heating/cooling features that cause small changes in the vertical velocity at the surface, is not propagated very far into the solution. A solution of the elliptic equation for w without any forcing term but with random noise at the surface was computed to show this property. Fig. 3 shows the random values of the vertical velocity at the surface. Fig. 4 and Fig. 5 show the resulting vertical velocity at the first and second levels of the model. Already at the second level the perturbations at the surface have been reduced by a factor of 10, i.e., the small scale irregularities at the lower boundary are not propagated very far into upper levels, exactly as expected from mathematical theory. This result should be contrasted with the sensitivity of the primitive equations at the surface that requires the ad hoc boundary layer treatment to prevent the rapid growth of perturbations at the surface.

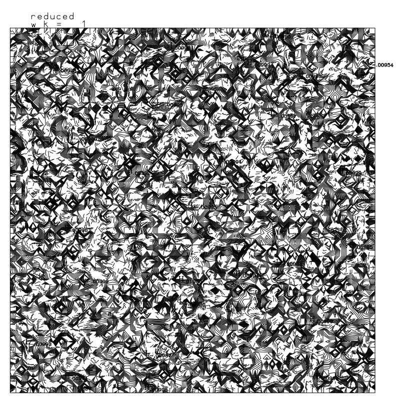



Figure 3: Random numbers for w at surface

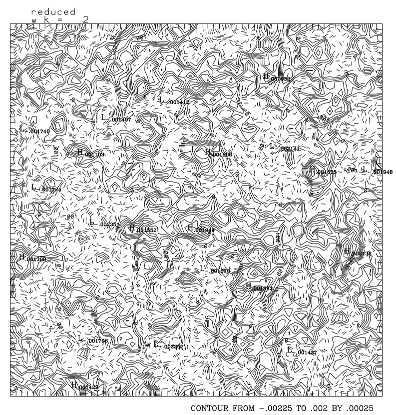



Figure 4: w at 1 km

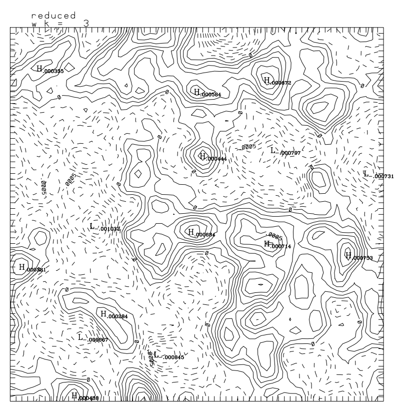



Figure 5: w at 2 km

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